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A Strategic Model of Social and Economic Networks*

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We study the stability and efficiency of social and economic networks, when self-interested individuals can form or sever links. First, for two stylized models, we characterize the stable and efficient networks. There does not always exist a stable network that is efficient. Next, we show that this tension persists generally: to assure that there exists a stable network that is efficient, one is forced to allocate resources to nodes that are not responsible for any of the production. We characterize one such allocation rule: the equal split rule, and another rule that arises naturally from bargaining of the players. *Journal Economic Literature Classification Numbers*: A14, D20, J00. © 1996 Academic Press, Inc.

1. INTRODUCTION

Network structures play an important role in the organization of some significant economic relationships. Informal social networks are often the means for communicating information and for the allocation of goods and services which are not traded in markets. Among such goods one can mention not only invitations to parties and other forms of exchanging friendship, but also information about job openings, business opportunities, and the like. In the context of a firm, the formal network through which relevant information is shared among the employees may have an important effect on the firm's productivity. In both contexts, the place of an

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agent in the network may affect not only his or her productivity, but also his or her bargaining position relative to others and this might be reflected in the design of such organizations.

The main goal of this paper is to begin to understand which networks are stable, when self-interested individuals choose to form new links or to sever existing links. This analysis is designed to give us some predictions concerning which networks are likely to form, and how this depends on productive and redistributive structures. In particular, we will examine the relationship between the set of networks which are productively efficient, and those which are stable. The two sets do not always intersect. Our analysis begins in the context of several stylized models, and then continues in the context of a general model.

This work is related to a number of literatures which study networks in a social science context. First, there is an extensive literature on social networks from a sociological perspective (see Wellman and Berkowitz [28] for one recent survey) covering issues ranging from the interfamily marriage structure in 15th century Florence to the communication patterns among consumers (see Iacobucci and Hopkins [11]). Second, occasional contributions to microeconomic theory have used network structures for such diverse issues as the internal organization of firms (e.g., Boorman [2], Keren and Levhari [16]), employment search (Montgomery [18]), systems compatibility (see Katz and Shapiro [15]), information transmission (Goyal [5]), and the structure of airline routes (Hendricks, *et al.* [7, 8], Starr and Stinchcombe [26]). Third, there is a formal game theoretic literature which includes the marriage problem and its extensions (Gale and Shapley [4], Roth and Sotomayor [24]), games of flow (Kalai and Zemel [14]), and games with communication structures (Aumann and Myerson [1], Kalai *et al.* [13], Kirman *et al.* and Myerson [19]). Finally, the operations research literature has examined the optimization of transportation and communications networks. One area of that research studies the allocation of costs on minimal cost spanning trees and makes explicit use of cooperative game theory. (See Sharkey [25] for a recent survey.)

The main contribution of this paper to these existing literatures is the modelling and analysis of the stability of networks when the nodes themselves (as individuals) choose to form or maintain links. The issue of graph endogeneity has been studied in specific contexts including cooperative games under the Shapley value (see Aumann and Myerson [1]) and the marriage problem (see Roth and Sotomayor [24]). The contribution here lies in the diversity and generality of our analysis, as well as in the focus on the tension between stability and efficiency.

Of the literatures we mentioned before, the one dealing with cooperative games that have communication structures is probably the closest in methodology to our analysis. This direction was first studied by Myerson [19], and then by Owen [22], van den Nouweland and Borm [21], and others (see van den Nouweland [20] for a detailed survey). Broadly speaking, the contribution of that literature is to model restrictions on coalition formation in cooperative games. Much of the analysis is devoted to some of the basic issues of cooperative game theory such as the characterization of value allocations with communication structures. Our work differs from that literature in some important respects. First, in our framework the value of a network can depend on exactly how agents are interconnected, not just who they are directly or indirectly connected to. Unlike games with communication, different forms of organization might generate different levels of profit or utility, even if they encompass (interconnect) exactly the same players. Second, we focus on network stability and formation and its relationship to efficiency. Third, an important aspect of our work is the application of this approach to some specific models of the organization of firms and network allocation mechanisms of non-market goods.

The paper proceeds as follows. In Section 2 we provide the definitions comprising the general model. In Section 3 we examine several specific versions of the model with stylized value functions. For each of these models we describe the efficient networks and the networks which are stable. We note several instances of incompatibility between efficiency and stability. In Section 4, we return to the general model to study means of allocating the total production or utility of a network. We examine in detail which types of allocation rules allow for stability of efficient networks. We conclude with a result characterizing the implications of equal bargaining power for allocation rules.

2. DEFINITIONS

Let $\mathcal{N} = \{1, ..., N\}$ be the finite set of players. The network relations among these players are formally represented by graphs whose *nodes* are identified with the players and whose arcs capture the pairwise relations.

2.1. *Graphs*

The complete graph, denoted g^N, is the set of all subsets of $\mathcal{N}$ of size 2. The set of all possible graphs on $\mathcal{N}$ is then $\{g \mid g \subset g^N\}$. Let ij denote the subset of $\mathcal{N}$ containing i and j and is referred to as the *link* ij. The interpretation is that if $ij \in g$, then nodes i and j are directly connected (sometimes referred to as adjacent), while if $ij \notin g$, then nodes i and j are not directly connected.[1]

Let $g+ij$ denote the graph obtained by adding link ij to the existing graph g and $g-ij$ denote the graph obtained by deleting link ij from the existing graph g (i.e., $g+ij=g\cup\{ij\}$ and $g-ij=g\backslash\{ij\}$).

Let $N(g)=\{i \mid \exists j \text{ s.t. } ij\in g\}$ and $n(g)$ be the cardinality of $N(g)$. A *path in g connecting* i_1 and i_n is a set of distinct nodes $\{i_1, i_2, ..., i_n\}\subset N(g)$ such that $\{i_1 i_2, i_2 i_3, ..., i_{n-1} i_n\}\subset g$.

The graph $g'\subset g$ is a *component of* g, if for all $i\in N(g')$ and $j\in N(g')$, $i\neq j$, there exists a path in g' connecting i and j, and for any $i\in N(g')$ and $j\in N(g)$, $ij\in g$ implies that $ij\in g'$.

2.2. *Values and Allocations*

Our interest will be in the total productivity of a graph and how this is allocated among the individual nodes. These notions are captured by a value function and an allocation function.

The *value of a graph* is represented by v: $\{g \mid g\subset g^N\}\to\mathbb{R}$. The set of all such functions is V. In some applications the value will be an aggregate of individual utilities or productions, $v(g)=\sum_i u_i(g)$, where u_i: $\{g \mid g\subset g^N\}\to\mathbb{R}$.

A graph $g\subset g^N$ is *strongly efficient* if $v(g)\geqslant v(g')$ for all $g'\subset g^N$. The term strong efficiency indicates maximal total value, rather than a Paretian notion. Of course, these are equivalent if value is transferable across players.

An *allocation rule* Y: $\{g \mid g\subset g^N\}\times V\to\mathbb{R}^N$ describes how the value associated with each network is distributed to the individual players. $Y_i(g, v)$ is the payoff to player i from graph g under the value function v.

2.3. *Stability*

As our interest is in understanding which networks are likely to arise in various contexts, we need to define a notion which captures the stability of a network. The definition of a stable graph embodies the idea that players have the discretion to form or sever links. The formation of a link requires the consent of both parties involved, but severance can be done unilaterally.

[1] The graphs analyzed here are non-directed. That is, it is not possible for one individual to link to another, without having the second individual also linked to the first. (Graphs where unidirectional links are possible are sometimes called digraphs.) Furthermore, links are either present or not, as opposed to having connections with variable intensities (a valued graph). See Iacobucci [10] for a detailed set of definitions for a general analysis of social networks. Such alternatives are important, but are beyond the scope of our analysis.

The graph g is *pairwise stable* respect to v and Y if

(i) for all $ij \in g$, $Y_i(g, v) \geqslant Y_i(g - ij, v)$ and $Y_j(g, v) \geqslant Y_j(g - ij, v)$ and

(ii) for all $ij \notin g$, if $Y_i(g, v) < Y_i(g + ij, v)$ then $Y_j(g, v) > Y_j(g + ij, v)$.

We shall say that g is *defeated* by g' if $g' = g - ij$ and (i) is violated for ij, or if $g' = g + ij$ and (ii) is violated for ij.

Condition (ii) embodies the assumption that, if i strictly prefers to form the link ij and j is just indifferent about it, then it will be formed.

The notion of pairwise stability is not dependent on any particular formation process. That is, we have not formally modeled the procedure through which a graph is formed. Pairwise stability is a relatively weak notion among those which account for link formation and as such it admits a relatively larger set of stable allocations than might a more restrictive definition or an explicit formation procedure. (See Section 5 for more discussion of this.) For our purposes, such a weak definition provides strong results, since in many instances it already narrows the set of graphs substantially.

There are many obvious modifications of the above definition which one might consider. One obvious strengthening would be to allow changes to be made by coalitions which include more than two players. To keep the presentation uncluttered, we will go through the analysis using only the stability notion defined above and relegate all the remarks on other variations to Section 5.

3. TWO SPECIFIC MODELS

We begin by analyzing several stylized versions of the general model outlined in the last section. There are innumerable versions which one can think of. The examples presented in this section are meant to capture some basic and diverse issues arising in social and economic networks. In particular, we illustrate what the application of pairwise stability predicts concerning which graphs might form and whether these are strongly efficient.

3.1. *The Connections Model*

This first example models social communication among individuals.[2] Individuals directly communicate with those to whom they are linked.

[2] Goyal [5] considers a related model. His is a non-cooperative game of one-sided link formation and it differs in some of the specifications as well, but it is close in terms of its flavor and motivation.

Through these links they also benefit from indirect communication from those to whom their adjacent nodes are linked, and so on. The value of communication obtained from other nodes depends on the distance to those nodes. Also, communication is costly so that individuals must weigh the benefits of a link against its cost.

Let $w_{ij} \geqslant 0$ denote the "intrinsic value" of individual j to individual i and c_{ij} denote the cost to i of maintaining the link ij. The utility of each player i from graph g is then

$$u_i(g) = w_{ii} + \sum_{j \neq i} \delta^{t_{ij}} w_{ij} - \sum_{j:\, ij \in g} c_{ij},$$

where t_{ij} is the number of links in the shortest path between i and j (setting $t_{ij} = \infty$ if there is no path between i and j), and $0 < \delta < 1$ captures the idea that the value that i derives from being connected to j is proportional to the proximity of j to i.[3] Less distant connections are more valuable than more distant ones, but direct connections are costly. Here

$$v(g) = \sum_{i \in \mathcal{N}} u_i(g).$$

3.1.1. *Strong Efficiency in the Connections Model*

In what follows we focus on the symmetric version of this model, where $c_{ij} = c$ for all ij and $w_{ii} = 1$ for all $j \neq i$ and $w_{ij} = 0$. The term *star* describes a component in which all players are linked to one central player and there are no other links: $g \subset g^N$ is a star if $g \neq \emptyset$ and there exists $i \in \mathcal{N}$ such that if $jk \in g$, then either $j = i$ or $k = i$. Individual i is the center of the star.

PROPOSITION 1. *The unique strongly efficient network in the symmetric connections model is*

(i) *the complete graph g^N if $c < \delta - \delta^2$,*

(ii) *a star encompassing everyone if $\delta - \delta^2 < c < \delta + ((N-2)/2)\, \delta^2$, and*

(iii) *no links if $\delta + ((N-2)/2)\, \delta^2 < c$.*

Proof. (i). Given that $\delta^2 < \delta - c$, any two agents who are not directly connected will improve their utilities, and thus the total value, by forming a link.

(ii) and (iii). Consider g', a component of g containing m individuals. Let $k \geqslant m - 1$ be the number of links in this component. The value of these direct links is $k(2\delta - 2c)$. This leaves at most $m(m-1)/2 - k$ indirect links.

[3] The shortest path is sometimes called the geodesic, and t_{ij} the geodesic distance.

The value of each indirect link is at most $2\delta^2$. Therefore, the overall value of the component is at most

$$k(2\delta - 2c) + (m(m-1) - 2k)\,\delta^2. \tag{1}$$

If this component is a star then its value would be

$$(m-1)(2\delta - 2c) + (m-1)(m-2)\,\delta^2. \tag{2}$$

Note that $(1)-(2) = (k-(m-1))(2\delta - 2c - 2\delta^2)$, which is at most 0 since $k \geqslant m-1$ and $c > \delta - \delta^2$, and less than 0 if $k > m-1$. The value of this component can equal the value of the star only when $k = m-1$. Any graph with $k = m-1$, which is not a star, must have an indirect connection which has a path longer than 2, getting value less than $2\delta^2$. Therefore, the value of the indirect links will be below $(m-1)(m-2)\,\delta^2$, which is what we get with star.

We have shown that if $c > \delta - \delta^2$, then any component of a strongly efficient graph must be a star. Note that any component of an strongly efficient graph must have nonnegative value. In that case, a direct calculation using (2) shows that a single star of $m+n$ individuals is greater in value than separate stars of m and n individuals. Thus if the strongly efficient graph is nonempty, it must consist of a single star. Again, it follows from (2) that if a star of n individuals has nonnegative value, then a star of $n+1$ individuals has higher value. Finally, to complete (ii) and (iii) note that a star encompassing everyone has positive value only when $\delta + ((N-2)/2)\,\delta^2 > c$. ∎

This result has some of the same basic intuition as the hub and spoke analysis of Hendricks *et al.* [8] and Starr and Stinchcombe [26], except that the values of graphs are generated in different manners.

3.1.2. *Stability in the Connections Model without Side Payments*

Next, we examine some implications of stability for the allocation rule $Y_i(g) = u_i(g)$. This specification might correspond best to a social network in which by convention no payments are exchanged for "friendship."

Proposition 2. *In the symmetric connections model with* $Y_i(g) = u_i(g)$:

(i) *A pairwise stable network has at most one (non-empty) component.*

(ii) *For* $c < \delta - \delta^2$, *the unique pairwise stable network is the complete graph,* g^N.

(iii) *For* $\delta - \delta^2 < c < \delta$, *a star encompassing all players is pairwise stable, but not necessarily the unique pairwise stable graph.*

(iv) *For $\delta < c$, any pairwise stable network which is non-empty is such that each player has at least two links and thus is inefficient.*[4]

Proof. (i) Suppose that g is pairwise stable and has two or more non-trivial components. Let u^{ij} denote the utility which accrues to i from the link ij, given the rest of g: so $u^{ij} = u_i(g + ij) - u_i(g)$ if $ij \notin g$ and $u^{ij} = u_i(g) - u_i(g - ij)$ if $ij \in g$. Consider $ij \in g$. Then $u^{ij} \geqslant 0$. Let kl belong to a different component. Since i is already in a component with j, but k is not, it follows that $u^{kj} > u^{ij} \geqslant 0$, since k will also receive δ^2 in value for the indirect connection to i, which is not included in u^{ij}. For similar reasons, $u^{jk} > u^{lk} \geqslant 0$. This contradicts pairwise stability, since $jk \notin g$.

(ii) It follows from the fact that in this cost range, any two agents who are not directly connected benefit from forming a link.

(iii) It is straightforward to verify that the star is stable. It is the unique stable graph in this cost range if $N = 3$. It is never the unique stable graph if $N = 4$. (If $\delta - \delta^3 < c < \delta$, then a line is also stable, and if $c < \delta - \delta^3$, then a circle[5] is also stable.)

(iv) In this range, pairwise stability precludes "loose ends" so that every connected agent has at least two links. This means that the star is not stable, and so by Proposition 1, any non-empty pairwise stable graph must be inefficient. ∎

Remark. The results of Proposition 2 would clearly still hold if one strengthens pairwise stability to allow for deviations by groups of individuals instead of just pairs. This would lean even more heavily on the symmetry assumption.

Remark. Part (iv) implies that in the high cost range (where $\delta < c$) the only non-degenerate networks which are stable are those which are over-connected from an efficiency perspective. (We will return to this tension between strong efficiency and stability later, in the analysis of the general model.) Since $\delta < c$, no individual is willing to maintain a link with another individual who does not bring additional new value from indirect connections. Thus, each node must have at least two links, or none. This means that the star cannot be stable: the center will not wish to maintain links with any of the end nodes.

The following example features an over-connected pairwise stable graph. The example is more complex than necessary (a circle with $N = 5$ will

[4] If $\delta + ((N-2)/2)\,\delta^2 > c$, then all pairwise stable networks are inefficient since then the empty graph is also inefficient.

[5] $g \subset g^N$ is a *circle* if $g \neq \varnothing$ and there exists $\{i_1, i_2, ..., i_n\} \subset \mathcal{N}$ such that $g = \{i_1 i_2, i_2 i_3, ..., i_{n-1} i_n, i_n i_1\}$.

illustrate the same point), but it illustrates that pairwise stable graphs can be more intricate than the simple stars and circles.

EXAMPLE 1. Consider the "tetrahedron" in Fig. 1. Here $N=16$. A star would involve 15 links and a total value of $30\delta+210\delta^2-30c$. The tetrahedron has 18 links and a total value of $36\delta+48\delta^2+60\delta^3+72\delta^4+24\delta^5-36c$, which (since $c>\delta$ and $\delta<1$) is less than that of the star.

Let us verify that the tetrahedron is pairwise stable. (Recall that u^{ij} denotes the utility which accrues to i from the link ij given the rest of g: so $u^{ij}=u_i(g+ij)-u_i(g)$ if $ij\notin g$ and $u^{ij}=u_i(g)-u_i(g-ij)$ if $ij\in g$.) Given the symmetry of the graph, the following inequalities assure pairwise stability of the graph: $u^{12}\geqslant 0$, $u^{21}\geqslant 0$, $u^{23}\geqslant 0$, $u^{13}\leqslant 0$, $u^{14}\leqslant 0$, $u^{15}\leqslant 0$, and $u^{26}\leqslant 0$. The first three inequalities assure that no one wants to sever a link. The next three inequalities assure that no new link can be improving to two agents if one of those agents is a "corner" agent. The last inequality assures that no new link can be improving to two agents if both of those agents are not "corner" agents. It is easy to check that $u^{21}>u^{12}$, $u^{23}>u^{12}$, $u^{13}<u^{14}$, $u^{15}<u^{14}$, and $u^{14}<u^{26}$. Thus we verify that $u^{12}\geqslant 0$ and $u^{26}\leqslant 0$:

$$u^{12}=\delta-\delta^8+\delta^2-\delta^7+\delta^3-\delta^6+2(\delta^4-\delta^5)-c,$$

$$u^{26}=\delta-\delta^5+\delta^2-\delta^4+\delta^2-\delta^5+2(\delta^3-\delta^4)-c,$$

If $c=1$ and $\delta=0.9$, then (approximately) $u^{12}=0.13$ and $u^{26}=-0.17$.

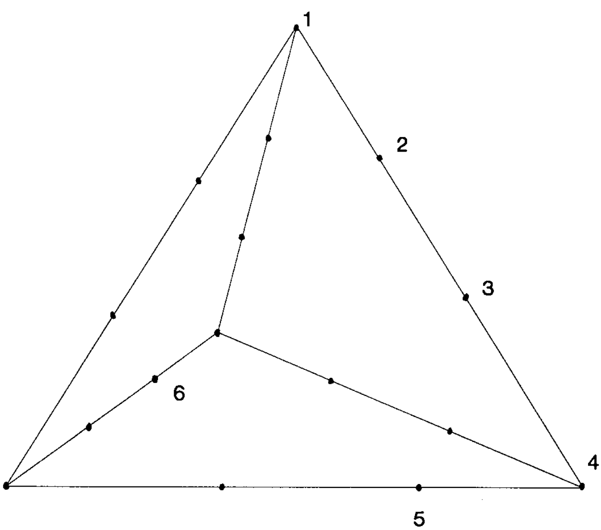

FIGURE 1

In this example, the graph is stable since each link connects an individual indirectly to other valuable individuals. The graph cannot be too dense, since it then becomes too costly to maintain links relative to their limited benefit. The graph cannot be too sparse as nodes will have incentives to add additional links to those links which are currently far away and/or sever current links which are not providing much value.

Before proceeding, we remark that the results presented for the connections model are easily adapted to replace $\delta^{t_{ij}}$ by any non-increasing function $f(t_{ij})$, by simply substituting $f(t_{ij})$ whenever $\delta^{t_{ij}}$ appears. One such alternative specification is a truncated connections model where players benefit only from connections which are not mode distant than some bound D. The case of $D=2$, for example, has the natural interpretation that i benefits from j only if they are directly connected or if they have a "mutual friend" to whom both are directly connected. It is immediate to verify that Propositions 1 and 2 continue to hold for the truncated connections models. In addition we have the following observations.

PROPOSITION 3. *In the truncated connections model with bound D*

(i) $t_{ij} \leqslant 2D-1$ *for all i and j which belong to a pairwise stable component.*

(ii) *For* $D=2$ *and* $\delta<c$ *no member in a pairwise stable component is in a position to disconnect all the paths connecting any two other players by unilaterally severing links.*

Proof. (i) Suppose $t_{ij} > 2D-1$. Consider one of the shortest paths between i and j. Let m belong to this path and $t_{mj}=1$. Note that $t_{ik} > D$, for any k such that j belongs to the shortest path between m and k and such that $t_{mk} \leqslant D$. This is because $t_{jk} \leqslant D-1$ and $t_{ij} > 2D-1$. Therefore, $u^{ij} > u^{mj}$ (the inequality is strict since u^{ij} includes the value to i of the connection to m which is not present in u^{mj}) so i wants to link directly to j. (Recall the notation u^{ij} from the proof of Proposition 2.) An analogous argument establishes that j wants to link directly to i.

(ii) Suppose that player i occupies such a position. Let j and k be such that i can unilaterally disconnect them and such that t_{jk} is the longest minimal path among all such pairs. Since by (i), $t_{jk} \leqslant 3$, there is at least one of them, say j, such that $t_{ij}=1$. But then i prefers to sever the link to j, since the maximality of t_{jk} implies that there is no h to whom i's only indirect connection passes through j (otherwise, $t_{hk} > t_{jk}$). ∎

There are obvious extensions to the connections model which seem quite interesting. For instance, one might have a decreasing value for each connection (direct or indirect) as the total amount of connectedness increases.

Also, if communication is modeled as occurring with some probability across each link, then one cares not only about the shortest path, but also about other paths in the event that communication fails across some link in the shortest path.[6]

3.1.3. *Stability in the Connections Model with Side Payments*

In the connections model with side payments, players are able to exchange money in addition to receiving the direct benefits from belonging to the network. The allocation rule will reflect these side payments which might be agreed upon in bilateral negotiations or otherwise. This version exposes another source of discrepancy between the strongly efficient and stable networks. Networks which produce high values might place certain players in key positions that will allow them to "claim" a disproportionate share of the total value. This is particularly true for the strongly efficient star-shaped network. This induces other players to form additional links that mitigate this power at the expense of reducing the total value. This consideration is illustrated by the following example.

EXAMPLE 2. Let $N=3$ and v be as in the basic connections model. The graph $g=\{12, 23\}$ is strongly efficient for $\delta-\delta^2<c<\delta$. Suppose that the allocation rule Y allocates the whole value of any graph to the players having links in the graph and reflects equal bargaining power in the sense that $Y_i(g, v)-Y_i(g-ij, v)=Y_j(g, v)-Y_j(g-ij, v)$ for all g, i, and j (we characterize this equal bargaining power rule in Theorem 4). Then $Y_1(g, v)=Y_3(g, v)=\delta+\frac{2}{3}\delta^2-c$ and $Y_2(g, v)=2\delta+\frac{2}{3}\delta^2-2c$.[7] That is, each of the peripheral players pays the center $\frac{1}{3}\delta^2$. In the alternative network $g'=\{12, 23, 31\}$ (the circle), $Y_1(g', v)=Y_2(g', v)=Y_3(g', v)=2\delta-2c$, and no side payments are exchanged. In the range $\delta-\frac{2}{3}\delta^2<c<\delta$ the strongly efficient network g is uniquely stable, but in the range $\delta-\delta^2<c<\delta-\frac{2}{3}\delta^2$ the inefficient network g' is the only stable one.

As mentioned above, the reason for the tension between efficiency and stability is the strong bargaining position of the center in g: when c is not too large, g is destabilized by the link between the peripheral players who increase their share at the expense of the center.

This version of the connections model can be adapted to discuss issues in the internal organization of firms. Consider a firm whose output depends

[6] Two such alternative models are discussed briefly in the appendix of Jackson and Wolinsky [12].

[7] To see this, note that $Y_1(g-23, v)=Y_2(g-23, v)=\delta-c$, $Y_3(g-23, v)=0$, and $Y_1(g-12, v)=0$, $Y_2(g-12, v)=Y_3(g-12, v)=\delta-c$. Then from equal bargaining power, we have that $Y_2(g, v)-(\delta-c)=Y_1(g, v)-0=Y_3(g, v)-0$. Then using the fact that $Y_1(g, v)+Y_2(g, v)+Y_3(g, v)=4\delta+2\delta^2-4c$, one can solve for $Y(g, v)$.

on the organization of the employees as a network. The network would capture here the structure of communication and hence coordination between workers. The nodes of the graph correspond to the workers. (For simplicity we exclude the owner from the graph, although it is not necessary for the result.) The total value of the firm's output, v, is as above. The allocation rule, Y, specifies the distribution of the total value between the workers (wages) and the firm (profit). It captures the outcome of wage bargaining within the firm, where labor contracts are not binding, and where the bargained wage of a worker is half the surplus associated with that worker's employment. The assumption built into this rule is that the position of a worker who quits cannot be filled immediately, so $Y_i(g-i, v)$ and $v(g-i) - \sum_{j \neq i} Y_j(g-i, v)$ are identified as the bargaining disagreement points of the worker and firm respectively (where $g-i$ denotes the graph which remains when all links including i are deleted). Thus,

$$Y_i(g, v) = Y_i(g-i, v) + \tfrac{1}{2}\left[v(g) - \sum_{j \neq i} Y_j(g, v) - Y_i(g-i, v) \right.$$
$$\left. - \left(v(g-i) - \sum_{j \neq i} Y_j(g-i, v) \right) \right].$$

If we think of the owner as external to the network, this Y is not balanced, as the firm's profit is $v - \sum_i Y_i$.[8]

EXAMPLE 3. Let $N = 3$ and v be as above. Assume $Y_i(g-i, v) = 0$ which means that a worker who quits is not paid. The graph $g = \{12, 23\}$ is strongly efficient for $\delta - \delta^2 < c < \delta$. Note that $Y_1(g, v) = Y_3(g, v) = \frac{2}{3}\delta + \frac{1}{2}\delta^2 - \frac{2}{3}c$ and $Y_2(g, v) = \frac{4}{3}\delta + \frac{1}{2}\delta^2 - \frac{4}{3}c$, leaving a profit of $\frac{4}{3}\delta + \frac{1}{2}\delta^2 - \frac{4}{3}c$ for the firm. Consider $g' = \{12, 23, 31\}$. Here $Y_1(g', v) = Y_2(g', v) = Y_3(g', v) = \frac{4}{3}\delta - \frac{4}{3}c$, leaving a profit of $2(\delta - c)$ for the firm.

In the range $\delta - \delta^2 < c < \delta - \frac{3}{4}\delta^2$ the network g is the strongly efficient form, but the network g' is more profitable to the firm, since it weakens the bargaining position of the worker occupying the center position in the graph g. This point complements existing work on internal wage bargaining and its consequences for the structure of firms. Stole and Zweibel (1993) investigate how internal wage bargaining distorts employment decisions, the extent of investment in capital, and the division of the workforce among activities (see also Grout [6] and Horn and Wolinsky [9]). The current framework adds explicitly the network organization of the firm.

[8] If the owner is included explicitly as a player, then Y coincides with the equal bargaining power rule examined in Section 4.

3.2. *The Co-author Model*

Here nodes are interpreted as researchers who spend time writing papers. Each node's productivity is a function of its links. A link represents a collaboration between two researchers. The amount of time a researcher spends on any given project is inversely related to the number of projects that researcher is involved in. Thus, in contrast to the connections model, here indirect connections will enter the utility function in a negative way as they detract from one's co-author time.

The fundamental utility or productivity of player i given the network g is

$$u_i(g) = \sum_{j:\, ij \in g} w_i(n_i, j, n_j) - c(n_i),$$

where $w_i(n_i, j, n_j)$ is the utility derived by i from a direct contact with j when i and j are involved in n_i and n_j projects, respectively, and $c(n_i)$ is the cost to i of maintaining n_i links.

We analyze a more specific version of this model where utility is given by the following expression. For $n_i > 0$,

$$u_i(g) = \sum_{j:\, ij \in g} \left[\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j}\right] = 1 + \left(1 + \frac{1}{n_i}\right) \sum_{j:\, ij \in g} \frac{1}{n_j},$$

and for $n_i = 0$, $u_i(g) = 0$. This form assumes that each researcher has a unit of time which they allocate equally across their projects. The output of each project depends on the total time invested in it by the two collaborators, $1/n_i + 1/n_j$, and on some synergy in the production process captured by the interactive term $1/n_i n_j$.

The interactive term is inversely proportional to the number of projects each author is involved with. Here there are no direct costs of connection. The cost of connecting with a new author is that the new link decreases the strength of the interaction term with existing links.[9]

PROPOSITION 4. *In this co-author model*: (i) *if N is even, then the strongly efficient network is a graph consisting of $N/2$ separate pairs, and* (ii) *a pairwise stable network can be partitioned into fully intraconnected components, each of which has a different number of members.* (*If m is the number of members of one such component and n is the next largest in size, then $m > n^2$.*)

[9] An alternative version the co-author model appears in the appendix of Jackson and Wolinsky [12].

Proof. To see (i), note that

$$\sum_{i \in N} u_i(g) = \sum_{i:\, n_i > 0} \sum_{j:\, ij \in g} \left[\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j} \right],$$

so that

$$\sum_{i \in N} u_i(g) \leqslant 2N + \sum_{i:\, n_i > 0} \sum_{j:\, ij \in g} \frac{1}{n_i n_j},$$

and equality can only hold if $n_i > 0$ for all i. To finish the proof of (i), note that $\sum_{i:\, n_i > 0} \sum_{j:\, ij \in g} 1/n_i n_j \leqslant N$, with equality only if $n_i = 1 = n_j$ for all i and j, and $3N$ is the value of $N/2$ separate pairs.

To see (ii), consider i and j who are not linked. It follows directly from the formula for $u_i(g)$ that i will strictly want to link to j if and only if

$$\frac{1}{n_j + 1} \left(1 + \frac{1}{n_i + 1} \right) > \left[\frac{1}{n_i} - \frac{1}{n_i + 1} \right] \sum_{k:\, k \neq j,\, ik \in g} \frac{1}{n_k},$$

(substitute 0 on the right-hand side if $n_i = 0$) which simplifies to

$$\frac{n_i + 2}{n_j + 1} > \frac{1}{n_i} \sum_{k:\, k \neq j,\, ik \in g} \frac{1}{n_k}. \qquad (*)$$

The following facts are then true of a pairwise stable network.

1. If $n_i = n_j$, then $ij \in g$.

We show that if $n_j \leqslant n_i$, then i would like to link to j. Note that $(n_i + 2)/(n_j + 1) > 1$ while the right-hand side of $(*)$ is at most 1 (the average of n_i fractions). Therefore, i would like to link to j.

2. If $n_h \leqslant \mathrm{Max}\{n_k \mid ik \in g\}$, then i wants to link to h.

Let j be such that $ij \in g$ and $n_j = \mathrm{Max}\{n_k \mid ik \in g\}$. If $n_i \geqslant n_j - 1$ then $(n_i + 2)/(n_h + 1) \geqslant 1$. If $(n_i + 2)/(n_h + 1) > 1$ then $(*)$ clearly holds for i's link to h. If $(n_i + 2)/(n_h + 1) = 1$, then it must be that $n_h \geqslant 2$ and so $n_j \geqslant 2$. This means that the right-hand side of $(*)$ when calculated for adding the link h will be strictly less than 1. Thus $(*)$ will hold. If $n_i < n_j - 1$, then $(n_i + 1)/n_j < (n_i + 2)/(n_j + 1) < (n_i + 2)/(n_h + 1)$. Since $ij \in g$, it follows from $(*)$ that

$$\frac{n_i + 1}{n_j} \geqslant \frac{1}{n_i - 1} \sum_{k:\, k \neq j,\, ik \in g} \frac{1}{n_k}.$$

Also

$$\frac{1}{n_i - 1} \sum_{k:\, k \neq j,\, ik \in g} \frac{1}{n_k} \geqslant \frac{1}{n_i} \sum_{k:\, ik \in g} \frac{1}{n_k}$$

since the extra element on the right-hand side is $1/n_j$ which is smaller than (or equal to) all terms in the sum. Thus $(n_i+2)/(n_h+1) > 1/n_i \sum_{k:\, ik\in g} 1/n_k$.

Facts 1 and 2 imply that all players with the maximal number of links are connected to each other and nobody else. (By 1 they must all be connected to each other. By 2, anyone connected to a player with a maximal number of links would like to connect to all players with no more than that number of links, and hence to all those with that number of links.) Similarly, all players with the next to maximal number of links are connected to each other and to nobody else, and so on.

The only thing which remains to be shown is that if m is the number of members of one (fully intraconnected) component and n is the next largest in size, then $m > n^2$. Note that for i in the next largest component not to be willing to hook to j in the largest component it must be that $(n_i+2)/n_j + 1 \leqslant 1/n_i$ (using $(*)$, since all nodes to which i is connected also have n_i connections). Thus $n_j + 1 \geqslant n_i(n_i+2)$. It follows that $n_j > n_i^2$. ∎

The combination of the efficiency and stability results indicates that stable networks will tend to be over-connected from an efficiency perspective. This happens because authors only partly consider the negative effect their new links have on the productivity of links with existing co-authors.

4. THE GENERAL MODEL

We now turn to analyzing the general model.

As we saw in Propositions 1 and 2, as well as in some of the examples in the previous section, efficiency and pairwise stability are not always compatible. That is, there are situations in which no strongly efficient graphs are pairwise stable. Does this persist in general? In other words, if we are free to structure the allocation rule in any way we like, is it possible to find one such that there is always at least one strongly efficient graph which is pairwise stable? The answer, provided in Theorem 1 below, depends on whether the allocation rule is balanced across components or is free to allocate resources to nodes which are not productive.

DEFINITION. Given a permutation $\pi: \mathcal{N} \to \mathcal{N}$, let $g^\pi = \{ij \mid i = \pi(k), j = \pi(l), kl \in g\}$. Let v^π be defined by $v^\pi(g^\pi) = v(g)$.[10]

DEFINITION. The allocation rule Y is *anonymous* if, for any permutation π, $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$.

[10] In the language of social networks, g^π and g are said to be isomorphic.

Anonymity states that if all that has changed is the names of the agents (and not anything concerning their relative positions or production values in some network), then the allocations they receive should not change. In other words, the anonymity of Y requires that the information used to decide on allocations be obtained from the function v and the particular g, and not from the label of an individual.

DEFINITION. An allocation rule Y is *balanced* if $\sum_i Y_i(g, v) = v(g)$ for all v and g.

A stronger notion of balance, component balance, requires Y to allocate resources generated by any component to that component. Let $C(g)$ denote the set of components of g. Recall that a component of g is a maximal connected subgraph of g.

DEFINITION. A value function v is *component additive* if $v(g) = \sum_{h \in C(g)} v(h)$.[11]

DEFINITION. The rule Y is *component balanced* if $\sum_{i \in N(h)} Y_i(g, v) = v(h)$ for every g and $h \in C(g)$ and component additive v.

Note that the definition of component balance only applies when v is component additive. Requiring it otherwise would necessarily contradict balance.

THEOREM 1. *If $N \geqslant 3$, then there is no Y which is anonymous and component balanced and such that for each v at least one strongly efficient graph is pairwise stable.*

Proof. Let $N = 3$ and consider (the component additive) v such that, for all i, j, and k, $v(\{ij\}) = 1$, $v(\{ij, jk\}) = 1 + \varepsilon$ and $v(\{ij, jk, ik\}) = 1$. Thus the strongly efficient networks are of the form $\{ij, jk\}$. By anonymity and component balance, $Y_i(\{ij\}, v) = 1/2$ and

$$Y_i(\{ij, jk, ik\}, v) = Y_k(\{ij, jk, ik\}, v) = \tfrac{1}{3}. \qquad (*)$$

Then pairwise stability of the strongly efficient network requires that $Y_j(\{ij, jk\}, v) \geqslant 1/2$, since $Y_j(\{ij\}, v) = 1/2$. This, together with component balance and anonymity, implies that $Y_i(\{ij, jk\}, v) = Y_k(\{ij, jk\}, v) \leqslant 1/4 + \varepsilon/2$. But this and $(*)$ contradict stability of the strongly efficient network when ε is sufficiently small ($< 1/6$), since then i and k would both

[11] This definition implicitly requires that the value of disconnected players is 0. This is not necessary. One can redefine components to allow a disconnected node to be a component. One has also to extend the definition of v so that it assigns values to such components.

gain from forming a link. This example is easily extended to $N>3$, by assigning $v(g)=0$ to any g which has a link involving a player other than player 1, 2 or 3. ∎

Theorem 1 says that there are value functions for which there is no anonymous and component balanced rule which supports strongly efficient networks as pairwise stable, even though anonymity and component balance are reasonable in many scenarios. It is important to note that the value function used in the proof is not at all implausible and is easily perturbed without upsetting the result.[12] Thus, one can make the simple observation that this conflict holds for an open set of value functions.

Theorem 1 does not reflect a simple nonexistence problem. We can find an anonymous and component balanced Y for which there always exists a pairwise stable network. To see a rule which is both component balanced and anonymous, and for which there always exists a pairwise stable network, consider $\bar{Y}$ which splits equally each component's value among its members. More formally, if v is component additive let $\bar{Y}_i(g, v)=v(h)/n(h)$ (recalling that $n(h)$ indicates the number of nodes in the component h) where $i \in N(h)$ and $h \in C(g)$,[13] and for any v that is not component additive let $\bar{Y}_i(g, v)=v(g)/N$ for all i. A pairwise stable graph for $\bar{Y}$ can be constructed as follows. For any component additive v find g by constructing components $h_1, ..., h_n$ sequentially, choosing h_i to maximize $v(h)/n(h)$ over all non-empty components which use only nodes not in $\bigcup_{j=1}^{i-1} N(h_j)$ (and setting $h_i=\varnothing$ if this value is always negative). The implication of Theorem 1 is that such a rule will necessarily have the property that, for some value functions, all of the networks which are stable relative to it are also inefficient.

The conflict between efficiency and stability highlighted by Theorem 1 depends both on the particular nature of the value function and on the conditions imposed on the allocation rule. This conflict is avoided if attention is restricted to certain classes of value functions, or if conditions on the allocation rule are relaxed. The following discussion will address each of these in turn. First, we describe a family of value functions for which this conflict is avoided. Then, we discuss the implications of relaxing the anonymity and component balance conditions.

Definition. A link ij is *critical* to the graph g if $g-ij$ has more components than g or if i is linked only to j under g.

[12] One might hope to rely on group stability to try to retrieve efficiency. However, group stability will simply refine the set of pairwise stable allocations. The result will still be true, and in fact sometimes there will exist no group stable graph.

[13] Use the convention that $n(\varnothing)=1$ and $i \in N(\varnothing)$ if i is not linked to any other node.

A critical link is one such that if it is severed, then the component that it was a part of will become two components (or one of the nodes will become disconnected). Let h denote a component which contains a critical link and let h_1 and h_2 denote the components obtained from h by severing that link (where it may be that $h_1 = \varnothing$ or $h_2 = \varnothing$.)

Definition. The pair (g, v) satisfies *critical link monotonicity* if, for any critical link in g and its associated components h, h_1, and h_2, we have that $v(h) \geqslant v(h_1) + v(h_2)$ implies that $v(h)/n(h) \geqslant \max[v(h_1)/n(h_1), v(h_2)/n(h_2)]$.

Consider again $\bar{Y}$ as defined above. The following is true.

Claim. *If g is strongly efficient relative to a component additive v, then g is pairwise stable for $\bar{Y}$ relative to v if and only if (g, v) satisfies critical link monotonicity.*

Proof. Suppose that g is strongly efficient relative to v and is pairwise stable for $\bar{Y}$ relative to v. Then for any critical link ij, it must be that i and j both do not wish to sever the link. This implies that $v(h)/n(h) \geqslant \max[v(h_1)/n(h_1), v(h_2)/n(h_2)]$. Next, suppose that g is strongly efficient relative to a component additive v and that the critical link condition is satisfied. We show that g is pairwise stable for $\bar{Y}$ relative to v. Adding or severing a non-critical link will only change the value of the component in question without changing the number of nodes in that component. By strong efficiency and component additivity, the value of this component is already maximal and so there can be no gain. Next consider adding or severing a critical link. Severing a critical link leads to no benefit for either node, since by strong efficiency and component additivity $v(h) \geqslant v(h_1) + v(h_2)$, which by the critical link condition implies that $v(h)/n(h) \geqslant \max[v(h_1)/n(h_1), v(h_2)/n(h_2)]$. By strong efficiency and component additivity, adding a critical link implies that $v(h) \leqslant v(h_1) + v(h_2)$ (where h_1 and h_2 are existing components and h is the new component formed by adding the critical link). Suppose to the contrary that g is not stable to the addition of the critical link. Then, without loss of generality it is the case that $v(h)/n(h) > v(h_1)/n(h_1)$ and $v(h)/n(h) \geqslant v(h_2)/n(h_2)$. Taking a convex combination of these inequalities (with weights $n(h_1)/n(h)$ and $n(h_2)/n(h)$) we find that $v(h) > v(h_1) + v(h_2)$, contradicting the fact that $v(h) \leqslant v(h_1) + v(h_2)$. ∎

To get some feeling for the applicability of the critical link condition, note that if a strongly efficient graph has no critical links, then the condition is trivially satisfied. This is true in Proposition 1 (i) and (iii), for instance. Note, also, that the strongly efficient graphs described in Proposition 1 (ii) and Proposition 4 (i) satisfy the critical link condition, even though they consist entirely of critical links. Clearly, the value function described in the proof of Theorem 1 does not satisfy the critical link condition.

Consider next the role of the anonymity and component balance conditions in the result of Theorem 1. The proof of Theorem 1 uses anonymity, but it can be argued that the role of anonymity is not central in that a weaker version of Theorem 1 holds if anonymity is dropped. A detailed statement of this result appears in Section 5. The component balance condition, however, is essential for the result of Theorem 1.

To see that if we drop the component balance condition the conflict between efficiency and stability can be avoided, consider the equal split rule ($Y_i(g, v) = v(g)/N$). This is not component balanced as all agents always share the value of a network equally, regardless of their position. This rule aligns the objectives of all players with value maximization and, hence, it results in strongly efficient graphs being pairwise stable. In what follows, we identify conditions under which the equal split rule is the only allocation rule for which strongly efficient graphs are pairwise stable. This is made precise as follows.

DEFINITION. The value function v is *anonymous* if $v(g^\pi) = v(g)$ for all permutations π and graphs g.

Anonymity of v requires that v depends only on the shape of g.

DEFINITION. Y is *independent of potential links* if $Y(g, v) = Y(g, w)$ for all graphs g and value functions v and w such that there exists $j \neq i$ so that v and w agree on every graph except $g + ij$.

Such an independence condition is very strong. It requires that the allocation rule ignore some potential links. However, many allocation rules, such as the equal split and the one based on equal bargaining power (Theorem 4 below), satisfy independence of potential links.

THEOREM 2. *Suppose that Y is anonymous, balanced, and independent of potential links. If v is anonymous and all strongly efficient graphs are stable, then $Y_i(g, v) = v(g)/N$, for all i and strongly efficient g's.*

Proof. If g^N is strongly efficient the result follows from the anonymity of v and Y. The rest of the proof proceeds by induction. Suppose that $Y_i(g, v) = v(g)/N$, for all i and strongly efficient g's which have k or more links. Consider a strongly efficient g with $k-1$ links. We must show that $Y_i(g, v) = v(g)/N$ for all i.

First, suppose that i is not fully connected under g and $Y_i(g, v) > v(g)/N$. Find j such that $ij \notin g$. Let w coincide with v everywhere except on $g + ij$ (and all its permutations) and let $w(g + ij) > v(g)$. Now, $g + ij$ is strongly efficient for w and so by the inductive assumption, $Y_i(g+ij, w) = w(g+ij)/N > v(g)/N$. By the independence of potential links (applied iteratively, first changing v only on $g+ij$, then on a permutation of $g+ij$, etc.), $Y_i(g, w) = Y_i(g, v) > v(g)/N$. Therefore, for $w(g+ij) - v(g)$ sufficiently small, $g+ij$ is defeated by g under w (since i profits from severing the link ij), although $g+ij$ is strongly efficient while g is not—a contradiction.

Next, suppose that i is not fully connected under g and that $Y_i(g, v) < v(g)/N$. Find j such that $ij \notin g$. If $Y_j(g, v) > v(g)/N$ we reach a contradiction as above. So $Y_j(g, v) \leqslant v(g)/N$. Let w coincide with v everywhere except on $g+ij$ (and all its permutations) where $w(g+ij) = v(g)$. Now, $g+ij$ is strongly efficient for w and hence, by the inductive assumption, $Y_i(g+ij, w) = Y_j(g+ij, w) = v(g)/N$. This and the independence of potential links imply that $Y_i(g+ij, w) = v(g)/N > Y_i(g, v) = Y_i(g, w)$ and $Y_j(g+ij, w) = v(g)/N \geqslant Y_j(g, v) = Y_j(g, w)$. But this is a contradiction, since g is strongly efficient for w but is unstable. Thus we have shown that for any strongly efficient g, $Y_i(g, v) = v(g)/N$ for all i which are not fully connected under g. By anonymity of v and Y (and total balance of Y), this is also true for i's which are fully connected. ▌

Remark. The proof of Theorem 2 uses anonymity of v and Y only through their implication that any two fully connected players get the same allocation. We can weaken the anonymity of v and Y and get a stronger version of Theorem 2. The allocation rule Y satisfies *proportionality* if for each i and j there exists a constant k_{ij} such that $Y_i(g, v)/Y_j(g, v) = k_{ij}$ for any g in which both i and j are fully connected and for any v. The new Theorem 2 would read: *Suppose Y satisfies proportionality and is independent of potential links. If all strongly efficient graphs are pairwise stable, then* $Y_i(g, v) = s^i v(g)$, *for all i, v, and g's which are strongly efficient relative to v, where* $s^i = Y_i(g^N, v)/v(g^N)$. The proof proceeds like that of Theorem 2 with s^i taking the place of $1/N$.

Theorem 2 only characterizes Y at strongly efficient graphs. If we require the right incentives holding at all graphs then the characterization is made complete.

DEFINITION. Y is *pairwise monotonic* if g' defeats g implies that $v(g') > v(g)$.

Pairwise monotonicity is more demanding than the stability of strongly efficient networks, and in fact it is sufficiently strong (coupled with anonymity, balance, and independence of potential links) to result in a unique allocation rule for anonymous v. That is, the result that $Y_i(g, v) = v(g)/N$ is obtained for all g, not just strongly efficient ones, providing the following characterization of the equal split rule.

THEOREM 3. *If Y is anonymous, balanced, independent of potential links, and pairwise monotonic, then $Y_i(g, v) = v(g)/N$, for all i, and g, and anonymous v.*

Proof. The theorem is proven by induction. By the anonymity of v and Y and $Y_i(g^N, v) = v(g^N)/N$. We show that if $Y_i(g, v) = v(g)/N$ for all g where g has at least k links, then this is true when g has at least $k-1$ links.

First, suppose that i is not fully connected under g and $Y_i(g, v) > v(g)/N$. Find j such that $ij \notin g$. Let w coincide with v everywhere except that $w(g+ij) > v(g)$. By the inductive assumption, $Y_i(g+ij, w) = w(g+ij)/N$. By the independence of potential links, $Y_i(g, w) = Y_i(g, v) > v(g)/N$. Therefore, for $w(g+ij) - v(g)$ sufficiently small $g+ij$ is defeated by g under w (since i profits from severing ij), while $w(g+ij) > w(g)$, contradicting pairwise monotonicity.

Next, suppose that i is not fully connected under g and that $Y_i(g, v) < v(g)/N$. Find j such that $ij \notin g$ If $Y_j(g, v) > v(g)/N$ we reach a contradiction as above. So $Y_j(g, v) \leqslant v(g)/N$. Let w coincide with v everywhere except on $g+ij$ where $w(g+ij) = v(g)$. By the inductive assumption, $Y_i(g+ij, w) = Y_j(g+ij, w) = w(g+ij)/N$. This and the independence of potential links imply that $Y_i(g+ij, w) = w(g+ij)/N = v(g)/N > Y_i(g, v) = Y_i(g, w)$ and $Y_j(g+ij, w) = w(g+ij)/N = v(g)/N \geqslant Y_j(g, v) = Y_j(g, w)$. This is a contradiction, since $w(g) = w(g+ij)$ but g is defeated by $g+ij$.

Thus we have shown that $Y_i(g, v) = v(g)/N$ for all i which are not fully connected under g. By anonymity of v and Y (and total balance of Y), this is also true for i's which are fully connected. ∎

Note that the equal split rule, $Y_i(g, v) = v(g)/N$, for all i and g, satisfies anonymity, balance, and pairwise monotonicity, and is independent of potential links. Thus a converse of the theorem also holds.

Theorem 1 documented a tension between pairwise stability and efficiency. If one wants to guarantee that efficient graphs are stable, then one has to violate component balance (as the equal split rule does). In some circumstances, the rule by which resources are allocated may not be subject to choice, but may instead be determined by some process, such as bargaining among the individuals in the network. We conclude with a characterization of allocation rules satisfying equal bargaining power.

DEFINITION. An allocation rule Y satisfies *equal bargaining power*[14] (EBP) if for all v, g, and $ij \in g$

$$Y_i(g, v) - Y_i(g-ij, v) = Y_j(g, v) - Y_j(g-ij, v).$$

[14] Such an allocation rule, in a different setting, is called the "fair allocation rule" by Myerson [19].

Under such a rule every i and j gain equally from the existence of their link relative to their respective "threats" of severing this link.

The following theorem is an easy extension of a result by Myerson [19].

THEOREM 4. *If v is component additive, then the unique allocation rule Y which satisfies component balance and EBP is the Shapley value of the following game $U_{v,g}$ in characteristic function form.*[15] *For each S, $U_{v,g}(S) = \sum_{h \in C(g|_S)} v(h)$, where $g|_S = \{ij \in g : i \in S \text{ and } j \in S\}$.*

Although Theorem 4 is easily proven by extending Myerson's [19] proof to our setting (see the Appendix for details), it is an important strengthening of his result. In his formulation a graph represents a communication structure which is used to determine the value of coalitions. The value of a coalition is the sum over the value of the subcoalitions which are those which are intraconnected via the graph. For example, the value of coalition $\{1, 2, 3\}$ is the same under graph $\{12, 23\}$ as it is under graph $\{12, 13, 23\}$. In our formulation the value depends explicitly on the graph itself, and thus the value of any set of agents depends not only on the fact that they are connected, but on exactly how they are connected.[16] In all of the examples we have considered so far, the shape of the graph has played an essential role in the productivity.

The potential usefulness of Theorem 4 for understanding the implications of equal bargaining power is that it provides a formula which can be used to study the stability properties of different organizational forms under various value functions. For example, the following corollary brings two implications.

COROLLARY. *Let Y be the equal bargaining power rule from Theorem 4, and consider a component balanced v and any g and $ij \in g$:*

$$\text{If, for all } g' \subset g, \quad v(g') \geqslant v(g' - ij), \quad \text{then} \quad Y_i(g, v) \geqslant Y_i(g - ij, v).$$

$$\text{If, for all } g' \subset g, \quad v(g') \geqslant v(g' + ij), \quad \text{then} \quad Y_i(g, v) \geqslant Y_i(g + ij, v).$$

This follows directly from inspection of the Shapley value formula.

The first line of the corollary means, for example, that if v is such that links are of diminishing marginal contribution, then stable networks will

[15] $Y_i(g, v) = SV_i(U_{v,g})$, where the Shapley value of a game U in characteristic function form is $SV_i(U) = \sum_{S \subset \mathcal{N} - i} (U(S+i) - U(S))\ \#S!\ (N - \#S - 1)!/N!$.

[16] The graph structure is still essential to Myerson's formulation. For instance, the value of the coalition $\{1, 3\}$ is not the same under graph $\{12, 23\}$ as it is under graph $\{12, 13, 23\}$, since agents 1 and 3 cannot communicate under the graph $\{12, 23\}$ when agent 2 is not present.

not be too sparse in the sense that a subgraph of the strongly efficient graph will not be stable. Thus, in some circumstances, the equal bargaining power rule will guarantee that strongly efficient graphs are pairwise stable. However, as we saw in Theorem 1 this will not always be the case.

5. DISCUSSION OF THE STABILITY NOTION

The notion of stability that we have employed throughout this paper is one of many possible notions. We have selected this notion, not because it is necessarily more compelling than others, but rather because it is a relatively weak notion that still takes into account both link severance and link formation (and provides sharp results for most of our analysis). The purpose of the following discussion is to consider the implications of modifying this notion. At the outset, it is clear that stronger stability notions (admitting fewer stable graphs) will just strengthen Theorems 1, 2, and 3 (as well as Propositions 2, 3, and 4). That is, stronger notions would allow the conclusions to hold under the same or even weaker assumptions. Some of the observations derived in the examples change, however, depending on how the stability notion is strengthened.

Let us now consider a few specific variations on the stability notion and comment on how the analysis is affected. First, let us consider a stronger stability notion that still allows only link severance by individuals and link formation by pairs, but implicitly allows for side payments to be made between two agents who deviate to form a new link.

The graph g' defeats g under Y and v (allowing for side payments) if either

(i) $g' = g - ij$ and $Y_i(g, v) < Y_i(g', v)$ or $Y_j(g, v) < Y_j(g', v)$, or

(ii) $g' = g + ij$ and $Y_i(g', v) + Y_j(g', v) > Y_i(g, v) + Y_j(g, v)$.

We then say that g is pairwise stable allowing for side payments under Y and v, if it is not defeated by any g' according to the above definition.

Note that in a pairwise stable network allowing for side payments payoffs are still described by Y rather than Y plus transfers. This reflects the interpretation that Y is the allocation to each agent when one includes the side payments that have already been made. The network, however, still has to be immune against deviations which could involve additional side payments. This interpretation introduces an asymmetry in the consideration of side payments since severing a link, (i), can be done unilaterally, and so the introduction of additional side payments will not change the incentives, while adding a link, (ii), requires the consent of two agents and additional side payments relative to the new graph may play a role.[17]

Under this notion of stability allowing for side payments, a version of Theorem 1 holds without the anonymity requirement.

THEOREM 1′. *If $N \geqslant 3$, then there is no Y which is component balanced and such that for each v no strongly efficient graph is defeated* (*when allowing for side payments*) *by an inefficient one.*

The proof is in the Appendix. As this version reproduces the impossibility result of Theorem 1 without the anonymity restriction on Y, it supports our earlier assertion that this result was not driven by the anonymity of Y, but rather by the component balance condition.

Stability with side payments also results in stronger versions of Theorems 2 and 3 which are included in the Appendix.

Another possible strengthening of the stability notion would allow for richer combinations of moves to threaten the stability of a network. Note that the basic stability notion we have considered requires only that a network be immune to one deviating action at a time. It is not required that a network be immune to more complicated deviations, such as a simultaneous severance of some existing links and an introduction of a new link by two players (which is along the lines of the stability notion used in studying the marriage problem). It is also not required that a network be immune to deviations by more than two players simultaneously. Actually, the notion of pairwise stability that we have employed does not even contemplate the severance of more than one link by a single player.

The general impact of such stronger stability notions would be to strengthen our results, with the possible complication that in some cases there may exist no stable network. As an example, reconsider the co-author model and allow any pair of players to simultaneously sever any set of their existing links. Based on Proposition 4(ii), we know that any graph that could be stable under such a new definition must have fully intraconnected components. However, now a pair of players can improve for themselves by simultaneously severing all their links, except the link joining them. It follows that no graph is stable.

A weaker version of the stability notion can be obtained by altering (ii) to require that both deviating players who add a link be strictly better off in order for a new graph to defeat an old one. The notion we have used requires that one player be strictly better off and the other be weakly better off. Most of our discussion is not sensitive to this distinction; however, the conclusions of Theorems 2 and 3 are, as illustrated in the following example. Let $N = \{1, 2, 3, 4\}$, $g = \{14, 23, 24, 34\}$, and consider v with $v(g) = 1$,

[17] The results still hold if (i) is also altered to allow for side payments.

$v(g')=1$ if g' is a permutation of g, and $v(g')=0$ for any other g'. Consider Y such that $Y_1(g', v)=1/8$ $Y_2(g', v)=Y_3(g', v)=1/4$ and $Y_4(g', v)=3/8$ if g' is a permutation of g, and $Y_i(g', v)=0$ otherwise. Specify $Y_i(g', w)=w(g')/N$ for $w \neq v$, except if g' is a permutation of g and w agrees with v on g and all its subgraphs, in which case set $Y_i(g', w)=Y_i(g', v)$. This Y is anonymous, balanced, and independent of potential links. However, it is clear that $Y_1(g, v) \neq v(g)/N$. To understand where Theorems 2 and 3 fail, consider $g'=g+12$ and w which agrees with v on all subgraphs of g but gives $w(g+12)=1$. Under the definition of stability that we have used in this paper, $g+12$ defeats g since player 1 is made better off and 2 is unchanged $(Y_1(g+12, w)=1/4=Y_2(g+12, w))$; however, under this weakened notion of stability $g+12$ does not defeat g.

One way to sort out the different notions of stability would be to look more closely at the non-cooperative foundations of this model. Specifications of different procedures for graph formation (e.g., an explicit non-cooperative game) and equilibria of those procedures would lead to notions of stability. Some of the literature on communication structures have taken this approach to graph formation (see, e.g., Aumann and Myerson [1], Qin [23], and Dutta *et al.* [3]). Let us make only one observation in this direction. Central to our notion of stability is the idea that a deviation can include two players who come together to form a new link. The concept of Nash equilibrium does not admit such considerations. Incorporating deviations by pairs (or larger groups) of agents might most naturally involve a refinement of Nash equilibrium which explicitly allows for such deviations, such as strong equilibrium, coalition-proof Nash equilibrium,[18] or some other notion which allows only for certain coalitions to form. This constitutes a large project which we do not pursue here.

APPENDIX

THEOREM 1′. *If $N \geqslant 3$, then there is no Y which is component balanced and such that for each v no strongly efficient graph is defeated by an inefficient one.*

Remark. In fact, it is not required that no strongly efficient graph be defeated by an inefficient one, but rather that there be some strongly

[18] One can try to account for the incentives of pairs by considering an extensive form game which sequentially considers the addition of each link and uses a solution such as subgame perfection (as in Aumann and Myerson [1]). See Dutta *et al.* [3] for a discussion of this approach and an alternative approach based on coalition-proof Nash equilibrium.

efficient graph which is not defeated by any inefficient one and such that any permutation of that graph which is also strongly efficient is not defeated by any inefficient one. This is clear from the following proof.

Proof. Let $N=3$ and consider the same v given in the proof of Theorem 1. (For all i, j, and k, $v(\{ij\})=1$, $v(\{ij, jk\})=1+\varepsilon$ and $v(\{ij, jk, ik\})=1$, where the strongly efficient networks are of the form $\{ij, jk\}$.) Without loss of generality, assume that $Y_1(\{12\}), v) \geqslant 1/2$ and $Y_2(\{23\}, v) \geqslant 1/2$. (Given the component balance, there always exists such a graph with some relabeling of players.) Since $\{12, 13\}$ cannot be defeated by $\{12\}$, it must be that $Y_1(\{12, 13\}, v) \geqslant 1/2$. It follows from component balance that $1/2+\varepsilon \geqslant Y_2(\{12, 13\}, v)+Y_3(\{12, 13\}, v)$. Since $\{12, 13\}$ cannot be defeated by $\{12, 13, 23\}$, it must be that

$$\begin{aligned} \tfrac{1}{2}+\varepsilon &\geqslant Y_2(\{12, 13\}, v)+Y_3(\{12, 13\}, v) \\ &\geqslant Y_2(\{12, 13, 23\}, v)+Y_3(\{12, 13, 23\}, v). \end{aligned} \tag{$*$}$$

Similarly

$$\begin{aligned} \tfrac{1}{2}+\varepsilon &\geqslant Y_1(\{12, 23\}, v)+Y_3(\{12, 23\}, v) \\ &\geqslant Y_1(\{12, 13, 23\}, v)+Y_3(\{12, 13, 23\}, v). \end{aligned} \tag{$**$}$$

Now note that adding $(*)$ and $(**)$ we get

$$\begin{aligned} &Y_2(\{12, 13\}, v)+Y_3(\{12, 13\}, v)+Y_1(\{12, 23\}, v)+Y_3(\{12, 23\}, v) \\ &\quad \geqslant Y_1(\{12, 13, 23\}, v)+Y_2(\{12, 13, 23\}, v)+2Y_3(\{12, 13, 23\}, v). \end{aligned}$$

By component balance, we rewrite this as

$$2+2\varepsilon-Y_1(\{12, 13\}, v)-Y_2(\{12, 23\}, v) \geqslant 1+Y_3(\{12, 13, 23\}, v).$$

Thus

$$Y_1(\{12, 13\}, v)+Y_2(\{12, 23\}, v) \leqslant 1+2\varepsilon.$$

Then since no strongly efficient graph is defeated by an inefficient one, we know that $Y_1(\{12, 13\}, v) \geqslant Y_1(\{12\}, v)$ and $Y_2(\{12, 23\}, v) \geqslant Y_2(\{23\}, v)$, and so

$$Y_1(\{12\}, v)+Y_2(\{23\}, v) \leqslant 1+2\varepsilon.$$

Since $Y_1(\{12\}, v) \geqslant 1/2$, we know that $Y_2(\{23\}, v) \leqslant 1/2 + 2\varepsilon$. Thus, by component balance

$$Y_3(\{23\}, v) \geqslant \tfrac{1}{2} - 2\varepsilon.$$

Since $\{13, 23\}$ cannot be defeated by $\{23\}$, it must be that $Y_3(\{13, 23\}, v) \geqslant 1/2 - 2\varepsilon$. It follows from component balance that $1/2 + 3\varepsilon \geqslant Y_1(\{13, 23\}, v) + Y_2(\{13, 23\}, v)$. Since $\{13, 23\}$ cannot be defeated by $\{12, 13, 23\}$, it must be that

$$\begin{aligned} \tfrac{1}{2} + 3\varepsilon &\geqslant Y_1(\{13, 23\}, v) + Y_2(\{13, 23\}, v) \\ &\geqslant Y_1(\{12, 13, 23\}, v) + Y_2(\{12, 13, 23\}, v). \end{aligned} \tag{***}$$

Adding (*), (*), and (***), we find that

$$\begin{aligned} \tfrac{3}{2} + 5\varepsilon \geqslant 2[&Y_1(\{12, 13, 23\}, v) + Y_2(\{12, 13, 23\}, v) \\ &+ Y_3(\{12, 13, 23\}, v)] = 2, \end{aligned}$$

which is impossible for $\varepsilon < 1/10$.

Again, this is easily extended to $N > 3$, by assigning $v(g) = 0$ to any g which has a link involving a player other than player 1, 2, or 3. ∎

Definition. The allocation rule Y is *continuous*, if for any g, and v and w that differ only on g and for any ε, there exists δ such that $|v(g) - w(g)| < \delta$ implies that $|Y_i(g, v) - Y_i(g, w)| < \varepsilon$ for all $i \in N(g)$.

Theorem 2′. *Suppose that Y is anonymous, balanced, continuous, and independent of potential links. If v is anonymous and no strongly efficient graph is defeated (allowing for side payments) by an inefficient one, then, $Y_i(g, v) = v(g)/N$, for all i and strongly efficient g's.*

Proof. If g^N is strongly efficient the result follows from the anonymity of v and Y. The rest of the proof proceeds by induction. Suppose that $Y_i(g, v) = v(g)/N$, for all i and strongly efficient g's which have k or more links. Consider a strongly efficient g with $k-1$ links. We must show that $Y_i(g, v) = v(g)/N$ for all i.

First, suppose that i is not fully connected under g and $Y_i(g, v) > v(g)/N$. Find j such that $ij \notin g$. Let w coincide with v everywhere except on $g + ij$ (and all its permutations) and let $w(g + ij) > v(g)$. Now, $g + ij$ is strongly efficient for w and so by the inductive assumption, $Y_i(g+ij, w) = w(g+ij)/N > v(g)/N$. By the independence of potential links (applied iteratively, first changing v only on $g+ij$, then on a permutation of $g+ij$, etc.), $Y_i(g, w) = Y_i(g, v) > v(g)/N$. Therefore, for $w(g+ij) - v(g)$ sufficiently small, $g+ij$ is defeated by g under w (since i profits from severing the link ij), although $g+ij$ is strongly efficient while g is not—a contradiction.

Next, suppose that i is not fully connected under g and that $Y_i(g, v) < v(g)/N$. Find j such that $ij \notin g$. If $Y_j(g, v) > v(g)/N$ we reach a contradiction as above. So $Y_j(g, v) \leqslant v(g)/N$. Let $\varepsilon < [v(g)/N - Y_i(g, v)]/2$ and let w coincide with v everywhere except on $g+ij$ (and all its permutations) and let $w(g+ij) = v(g) + \delta/2$ where δ is the appropriate $\delta(\varepsilon)$ from the continuity definition. Now, $g+ij$ is strongly efficient for w and hence, by the inductive assumption, $Y_i(g+ij, w) = Y_j(g+ij, w) = [v(g) + \delta/2]/N$. Define u which coincides with v and w everywhere except on $g+ij$ (and all its permutations) and let $u(g+ij) = w(g) - \delta/2$. By the continuity of Y, $Y_i(g+ij, u) \geqslant v(g)/N - \varepsilon$ and $Y_j(g+ij, u) \geqslant v(g)/N - \varepsilon$. Thus, we have reached a contradiction, since g is strongly efficient for u but defeated by $g+ij$ since $Y_i(g+ij, u) + Y_j(g+ij, u) \geqslant 2v(g)/N - 2\varepsilon > 2v(g)/N - [v(g)/N - Y_i(g, v)] \geqslant Y_i(g, u) + Y_j(g, u)$. Thus we have shown that for a strongly efficient g, $Y_i(g, v) = v(g)/N$ for all i which are not fully connected under g. By anonymity of v and Y (and total balance of Y), this is also true for i's which are fully connected. ∎

Remark. The definition of "defeats" allows for side payments in (ii), but not in (i). To be consistent, (i) could be altered to read $Y_i(g', v) + Y_j(g', v) > Y_i(g, v) + Y_j(g, v)$, as side payments can be made to stop an agent from severing a link. Theorem 2 is still true. The proof would have to be altered as follows. Under the new definition (i) the cases $ij \notin g$ and $Y_i(g, v) + Y_j(g, v) > 2v(g)/N$ or $Y_i(g, v) + Y_j(, v) < 2v(g)/N$ would follow roughly the same lines as currently is used for the case where $ij \notin g$, and $Y_i(g, v) < v(g)/N$ and $Y_j(g, v) \leqslant v(g)/N$. (For $Y_i(g, v) + Y_j(g, v) > 2v(g)/N$ the argument would be that ij would want to sever ij from $g+ij$ when $g+ij$ is strongly efficient.) Then note that it is not possible that for all $ij \notin g$, $Y_i(g, v) + Y_j(g, v) = 2v(g)/N$, without having only two agents ij who are not fully connected, in which case anonymity requires that they get the same allocation, or by having $Y_i = v(g)/N$ for all i which are not fully connected.

Theorem 2 only characterizes Y at strongly efficient graphs. If we require the right incentives holding at all graphs then the characterization is made complete:

Definition. Y is *pairwise monotonic allowing for side payments* if g' defeats (allowing for side payments) g implies that $v(g') \geqslant v(g)$.

THEOREM 3′. *If Y is anonymous, balanced, independent of potential links, and pairwise monotonic allowing for side payments, then $Y_i(g, v) = v(g)/N$, for all i, and g, and anonymous v.*

Proof. The theorem is proven by induction. By the anonymity of v and Y and $Y_i(g^N, v) = v(g^N)/N$. We show that if $Y_i(g, v) = v(g)/N$ for all g where g has at least k links, then this is true when g has at least $k-1$ links.

First, suppose that i is not fully connected under g and $Y_i(g, v) > v(g)/N$. Find j such that $ij \notin g$. Let w coincide with v everywhere except that $w(g+ij) > v(g)$. By the inductive assumption, $Y_i(g+ij, w) = w(g+ij)/N$. By the independence of potential links, $Y_i(g, w) = Y_i(g, v) > v(g)/N$. Therefore, for $w(g+ij, w) - v(g)$ sufficiently small $g+ij$ is defeated by g under w (since i profits from severing ij), while $w(g+ij) > w(g)$, contradicting pairwise monotonicity.

Next, suppose that i is not fully connected under g and that $Y_i(g, v) < v(g)/N$. Find j such that $ij \notin g$. If $Y_j(g, v) > v(g)/N$ we reach a contradiction as above. So $Y_j(g, v) \leqslant v(g)/N$. Let w coincide with v everywhere except that $w(g+ij) < v(g)$ and $v(g)/N - w(g+ij)/N < \frac{1}{2}(v(g)/N - Y_i(g, v))$. Thus $2w(g+ij)/N > v(g)/N + Y_i(g, v)) \geqslant Y_j(g, v)) + Y_i(g, v))$. By the inductive assumption, $Y_i(g+ij, w) = Y_j(g+ij, w) = w(g+ij)/N$. Thus, we have reached a contradiction, since $w(g) > w(g+ij)$ but g is defeated by $g+ij$ since $Y_i(g+ij, w) + Y_j(g+ij, w) > Y_i(g, w) + Y_j(g, w)$.

Thus we have shown that $Y_i(g, v) = v(g)/N$ for all i which are not fully connected under g. By anonymity of v and Y (and total balance of Y), this is also true for i's which are fully connected. ∎

Proof of Theorem 4. Myerson's [19] proof shows that there is a unique Y which satisfies equal bargaining power (what he calls fair, having fixed our v) and such that $\sum Y_i$ is a constant across i's in any connected component when other components are varied (which is guaranteed by our component balance condition).

We therefore have only to show that $Y_i(g, v) = SV_i(U_{v,g})$ (as defined in the footnote below Theorem 4) satisfies component balance and equal bargaining power.

Fix g and define Y^g by $Y^g(g') = SV(U_{v, g \cap g'})$. (Note that $U_{v, g \cap g'}$ substitutes for what Myerson calls v/g'. With this in mind, it follows from Myerson's proof that Y^g satisfies equal bargaining power and that for any connected component h of g, $\sum_{i \in h} Y^g_i(g) = U_{v,g}(N(h))$. Since $Y^g(g) = Y(g)$, this implies that $\sum_{i \in h} Y^g_i(g) = U_{v,g}(N(h)) = v(h)$, so that Y satisfies component balance. Also, since Y^g satisfies equal bargaining power, we have that $Y^g_i(g) - Y^g_i(g-ij) = Y^g_j(g) - Y^g_j(g-ij)$. Now, $Y^g_i(g-ij) = SV_i(U_{v, g \cap g - ij}) = SV_i(U_{v, g-ij}) = Y_i(g-ij)$. Therefore, $Y_i(g) - Y_i(g-ij) = Y_j(g) - Y_j(g-ij)$, so that Y satisfies equal bargaining power as well.

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